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ON THE HAUSDORFF DIMENSION OF FIBRES

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ABSTRACT

It is well known that there are planar sets of Hausdorff dimension greater than 1 which are graphs of functions, i.e., all their vertical fibres consist of 1 point. We show this phenomenon does not occur for sets constructed in a certain "regular" fashion. Specifically, we consider sets obtained by partitioning a square into 4 subsquares, discarding 1 of them and repeating this on each of the 3 remaining squares, etc.; then almost all vertical fibres of a set so obtained have Hausdorff dimension at least $\frac{1}{2}$. Sharp bounds on the dimensions of sets of exceptional fibres are presented.

§1. Introduction

Consider the ensemble $\mathcal{T}$ of subsets F of the unit square constructed as follows (see Fig. 1). Partition the unit square into four congruent subsquares, and discard one of them. Apply the same operation, appropriately scaled, to each of the three remaining squares, with no constraints on the relative positions of the four discarded subsquares. Repeat this operation ad infinitum, obtaining in the limit a set $F \in \mathcal{T}$. (For a precise definition, see section 3.)

H. Furstenberg conjectured [private communication] that for *all* $F \in \mathcal{T}$, "most" *fibres*

$$F_x = \{y \in [0,1] \mid (x,y) \in F\} \tag{1.1}$$

have positive Hausdorff dimension. It is well known that every $F \in \mathcal{T}$ has dimension $(\log 3)/(\log 2)$, but this does not imply the conjecture—see section 2. The object of this note is to prove the following:

THEOREM 1. *For all* $F \in \mathcal{T}$, $\dim(F_x) \geq \frac{1}{2}$, *for almost all* $x \in [0,1]$ *with respect to Lebesgue measure.*

(In this paper, dim denotes Hausdorff dimension.)

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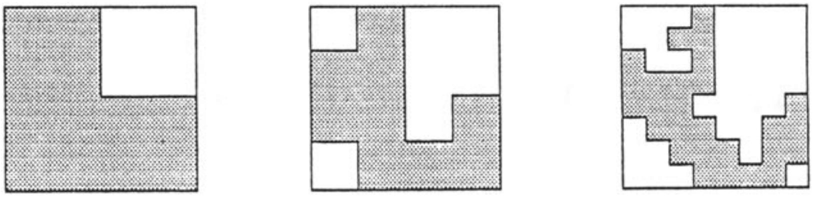

Fig. 1.

How large can the set of exceptional fibres be?

THEOREM 2. (i) *For all $F \in \mathcal{F}$ and $0 \le \alpha \le \frac{1}{2}$ we have*

$$\dim\{x \in [0,1] \mid \dim(F_x) \le \alpha\} \le h(\alpha) \tag{1.2}$$

where

$$h(\alpha) = -\alpha \log_2 \alpha - (1-\alpha)\log_2(1-\alpha), \qquad h(0) = 0 \tag{1.3}$$

is the binary entropy function.

In particular,

$$\dim\{x \mid \dim F_x = 0\} = 0.$$

(ii) *There exists $F^* \in \mathcal{F}$ for which*

$$\dim\{x \mid \dim(F_x^*) = \alpha\} = h(\alpha), \qquad \textit{for all } 0 \le \alpha \le 1. \tag{1.4}$$

The point of these results is that they hold for *all* $F \in \mathcal{F}$. The situation for "almost all" F has been studied extensively ([DG],[Fa],[MW]). It turns out that if $F \in \mathcal{F}$ is chosen "at random" then, with probability 1, almost all fibres have dimension log(3/2)/log 2. See Proposition 3 in section 2 for a precise statement. All these results are proved in section 3, after the necessary background is discussed in section 2. Section 4 contains an interpretation of Theorem 1 in terms of random walks on coloured trees.

§2. Background and further results

The dimensions of a fibre of a plane set in a random direction was determined by Marstrand [M]. In particular, his result implies the following. Let $F \in \mathcal{F}$ and $\gamma = \dim(F) = \log 3/\log 2$. Denote by $l(a,\theta)$ the line through the point a in direction θ. Then with probability 1,

$$\dim(F \cap l(a,\theta)) = \gamma - 1, \tag{2.1}$$

where $a \in F$ is chosen according to normalized γ-dimensional Hausdorff measure on F, and θ is chosen independently and uniformly. For fibres in a *fixed* direction, an example of Mattila [Mat., example 7.1] shows there exists a plane set of Hausdorff dimension 2, with *all* its fibres in one direction consisting of at most one point. Self-similar fractals are known to behave more regularly.

DEFINITION. The Minkowski dimension $\mathrm{Mdim}(E)$ of $E \subset [0,1]$ is defined by

$$\mathrm{Mdim}(E) = \limsup_{N\to\infty} \frac{1}{\log N}\left(\log \#\left\{1 \le j \le N \,\middle|\, E \cap \left[\frac{j-1}{N}, \frac{j}{N}\right] \neq \varnothing\right\}\right). \tag{2.2}$$

This quantity appears, in different guises, under the names "box dimension," "fractal index," etc. For any Borel set $E \subset [0,1]$, it is immediate that

$$\dim(E) \le \mathrm{Mdim}(E) \tag{2.3}$$

but strict inequality may occur.

In [F1], it is shown that equality holds in (2.3) for compact sets invariant under an endomorphism of the circle. Recently F. Ledrappier [private communication] showed that for compact sets invariant under certain toral endomorphisms, the Hausdorff dimension and Minkowski dimension coincide for almost all their fibres in a *fixed* direction. See [KP] for applications of this result. In our case the situation is different.

PROPOSITION 3. *Denote* $\gamma = \log 3/\log 2$.

(i) *For all* $F \in \mathcal{T}$

$$\tfrac{1}{2} \le \dim(F_x) \le \mathrm{Mdim}(F_x) \le \gamma - 1, \tag{2.4}$$

for almost all x with respect to Lebesgue measure m.

(ii) *Select a random $F \in \mathcal{T}$, by choosing randomly, independently, and with equal probabilities, which subsquare to discard from every square appearing in the construction of F.*

Then with probability 1,

$$\dim(F_x) = \mathrm{Mdim}(F_x) = \gamma - 1 \qquad \textit{for a.e. } x[m].$$

(iii) *There exists $F^* \in \mathcal{T}$ for which*

$$\dim(F^*_x) = \mathrm{Mdim}(F^*_x) = \tfrac{1}{2} \qquad \textit{for almost all } x[m].$$

(iv) *There exists $\tilde{F} \in \mathcal{T}$ such that*

$$\dim(\tilde{F}_x) = \tfrac{1}{2}, \qquad \mathrm{Mdim}(\tilde{F}_x) = \gamma - 1$$

for almost all $x[m]$.

The example in part (iv) of the proposition exhibits the maximal gap which can occur between the Hasdorff dimension and Minkowski dimension for *almost all* fibres of $F \in \mathfrak{F}$. On individual fibres the disparity can be greater.

PROPOSITION 4. *Given $F \in \mathfrak{F}$, consider the following "greedy algorithm" for selecting a fibre F_{x^*} of F. Choose the binary digits of x^* successively, so as to maximize the number of squares of each size in F which intersect F_{x^*}. Then*

$$\text{(2.5)} \qquad \textit{for all } F \in \mathfrak{F}, \qquad \mathrm{Mdim}(F_{x^*}) \geq \frac{\log(3/2)}{\log 2} \quad (\textit{clearly}),$$

but

$$\text{(2.6)} \qquad \textit{for some } F \in \mathfrak{F}, \qquad F_{x^*} \textit{ is countable while } \mathrm{Mdim}(F_{x^*}) = 1.$$

For a more precise description of the algorithm and for a proof of the proposition, see section 4.

We shall need some tools from dimension theory. The first is an extension, due to Davies, of a classical lemma of Frostman (see [C, chapter II]). Let H_β denote β-dimensional Hausdorff measure.

FROSTMAN'S LEMMA. *If $B \subset [0,1]$ is a Borel set for which $H_\beta(B) > 0$, then there exists a probability measure μ carried by a compact subset of B and satisfying*

$$\text{(2.7)} \qquad \mu([a,b]) \leq |b-a|^\beta \qquad \textit{for } [a,b] \subset [0,1].$$

The second tool is a lemma of Billingsley (see [B] or [Y]) slightly adapted to our purpose.

(2.8) Denote by $C_n(y)$ the binary interval $\left[\frac{j-1}{2^n}, \frac{j}{2^n}\right)$ containing y

where $j = \lfloor 2^n y \rfloor$.

BILLINGSLEY'S LEMMA. *Let ν be a probability measure on $[0,1]$. Assume $\Lambda \subset [0,1]$ is a Borel set satisfying*

$$\text{(2.9)} \qquad \nu(\Lambda) > 0 \quad \textit{and} \quad \Lambda \subset \left\{ y \,\middle|\, \liminf_{n\to\infty} \frac{-\log \nu(C_n(y))}{n \log 2} \geq \alpha \right\}.$$

Then $\dim(\Lambda) \geq \alpha$.

Finally, to compute the dimensions in Theorem 2 we need the following fact.

LEMMA 5. [M] *For any compact plane set F*, $\dim(F_x)$ *is a Borel measurable function of x.*

PROOF. We know that $\dim(F_x) < \alpha$ iff for some rational $q < \alpha$, for all rational $\epsilon > 0$ there exist $n \geq 1$ and rational open intervals $I_1, \ldots, I_n$ such that $\sum_{j=1}^n |I_j|^q < \epsilon$ and

$$F_x \subset \bigcup_{j=1}^n I_j. \tag{2.10}$$

Since the set of points x satisfying (2.10) is open, it follows that $\{x \mid \dim F_x < \alpha\}$ is a Borel set. ∎

§3. Proofs

We first formalize the definition of $\mathfrak{T}$. Let Φ be the collection of functions

$$\varphi : \bigcup_{n=0}^{\infty} \{0,1\}^n \times \{0,1\}^n \to \{0,1\}^2.$$

Define

$$F(\varphi) = \{(x,y) \mid \forall n \geq 1 \; x_n, y_n \in \{0,1\}, \\ \varphi(x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1}) \neq (x_n, y_n)\}, \tag{3.1}$$

where $x = \sum_1^\infty x_n 2^{-n}$ and similarly for y.

Then

$$\mathfrak{T} = \{F(\varphi) \mid \varphi \in \Phi\}. \tag{3.2}$$

Notice that when computing the dimension of the *vertical* fibres F_x for $F \in \mathfrak{T}$, we may assume that at all stages of the construction of F, the subsquare discarded is one of the two "top" ones. In other words, letting

$$\mathfrak{T}_0 = \{F(\varphi) \mid \varphi \in \Phi, \; \forall n \; \varphi(x_1, \ldots, x_n, y_1, \ldots, y_n) = (*, 1)\} \tag{3.3}$$

it is easily seen that for *any* $F \in \mathfrak{T}$, some $\tilde{F} \in \mathfrak{T}_0$ satisfies

$$\dim(\tilde{F}_x) = \dim(F_x) \qquad \text{for all } x \in [0,1] \tag{3.4}$$

(see [F2]).

Given $F = F(\varphi) \in \mathfrak{T}_0$ and $x = \sum_1^\infty x_n 2^{-n}$, $z = \sum_1^\infty z_n 2^{-n}$ which are not binary rationals, denote

(3.5)
$$\pi_x(z) = \sum_{n=1}^{\infty} y_n 2^{-n}$$

where y_n are defined inductively:

(3.6) σ_1 is defined by $\varphi(\phi) = (\sigma_1, 1)$, and then

$$y_1 = \begin{cases} 0, & \sigma_1 = x_1, \\ z_1, & \sigma_1 \neq x_1. \end{cases}$$

If $y_1, \ldots, y_{n-1}$ are already defined, then $\sigma_n = \sigma_n(x_1, \ldots, x_{n-1}; z_1, \ldots, z_{n-1})$ is determined by

(3.7)
$$\varphi(x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1}) = (\sigma_n, 1)$$

and

$$y_n = \begin{cases} 0, & \sigma_n = x_n, \\ z_n, & \sigma_n \neq x_n. \end{cases}$$

Finally, let

(3.8)
$$\pi(x,z) = (x, \pi_x(z)).$$

Note that if m denotes Lebesgue measure on [0,1], then $m\pi_x^{-1}$ is a measure on the fibre F_x, and $(m \times m)\pi^{-1}$ is a measure carried by F.

PROOF OF THEOREM 1. Choose $(x,z) \in [0,1]^2$ randomly according to Lebesgue measure $m \times m$. Define $y = \pi_x(z)$ and $\{\sigma_n\}$ as in (3.7). By the definition of π_x,

(3.9)
$$m\pi_x^{-1}(C_n(y)) = 2^{-\sum_{k=1}^{n} \sigma_k \oplus x_k}$$

($\oplus$ denotes sum mod 2).

Now σ_k is a function of $(x_1, \ldots, x_{k-1}, z_1, \ldots, z_{k-1})$ so the random variables $\{\sigma_k \oplus x_k\}_{k=1}^{\infty}$ are independent unbiased bits. Thus

(3.10)
$$\frac{1}{n} \sum_{k=1}^{n} (\sigma_k \oplus x_k) \to \tfrac{1}{2} \qquad \text{for a.e. } (x,z) \in [0,1]^2.$$

Using (3.9), this means that for a.e. $x[m]$ the set

(3.11)
$$\Lambda(x) = \left\{ y \in F_x \;\middle|\; \frac{1}{n \log 2} [-\log m\pi_x^{-1}(C_n(y))] \overset{n\to\infty}{\to} \frac{1}{2} \right\}$$

satisfies

$$m\pi_x^{-1}(\Lambda(x)) = 1. \tag{3.12}$$

Billingsley's lemma implies

$$\dim(\Lambda(x)) = \frac{1}{2} \qquad \text{for a.e. } x[m] \tag{3.13}$$

which proves Theorem 1. ∎

Proof of Theorem 2. (i) Fix $0 \le \alpha \le \frac{1}{2}$ and $F \in \mathfrak{T}$. Let

$$\Omega(\alpha) = \{x \in [0,1] \,|\, \dim(F_x) \le \alpha\}. \tag{3.14}$$

For any $\beta < \dim(\Omega(\alpha))$, Frostman's lemma provides a probability measure μ satisfying

$$\mu(\Omega(\alpha)) = 1, \qquad \mu[a,b] \le |b-a|^\beta \quad \text{for } [a,b] \subset [0,1]. \tag{3.15}$$

Thus part (i) of the theorem is a consequence of the following

Claim. If μ is a probability measure on [0,1], satisfying for some $\beta > h(\alpha)$

$$\mu[a,b] \le |b-a|^\beta \qquad \text{for all } [a,b] \subset [0,1], \tag{3.16}$$

then

$$\mu(\Omega(\alpha)) = 0. \tag{3.17}$$

Proof of Claim. Using the notation preceding Theorem 1, denote

$$B = \left\{ (x,z) \in [0,1]^2 \,\middle|\, \liminf_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (\sigma_k \oplus x_k) \le \alpha \right\} \tag{3.18}$$

(where $\sigma_k = \sigma_k(x_1,\ldots,x_{k-1},z_1,\ldots,z_{k-1})$). For every $z \in [0,1]$ the map

$$x \to \sum_{k=1}^{\infty} \epsilon_k 2^{-k}, \qquad \epsilon_k = x_k \oplus \sigma_k, \tag{3.19}$$

preserves Lebesgue measure and therefore also Hausdorff dimension (see the proof of Theorem 1). Consequently for fixed z

$$\dim\{x \,|\, (x,z) \in B\} = \dim\left\{ \sum_{k=1}^{\infty} \epsilon_k 2^{-k} \,\middle|\, \epsilon_k \in \{0,1\}, \liminf_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \epsilon_k \le \alpha \right\}. \tag{3.20}$$

It is well-known fact, due essentially to Besicovitch, that the set on the right-hand side of (3.20) has dimension $h(\alpha)$. See [Caj, example 8.2] and the references therein. Since $\beta > h(\alpha)$, the definition of Hausdorff measure and (3.16) imply that for any fixed z,

$$\mu\{x \mid (x,z) \in B\} = 0. \tag{3.21}$$

By Fubini

$$(\mu \times m)(B) = 0 \tag{3.22}$$

so that

$$m\pi_x^{-1}\{y \mid (x,y) \in \pi(B)\} = 0 \qquad \text{for a.e. } x[\mu]. \tag{3.23}$$

Now by the definition (3.18) of B and formula (3.9) we know

$$(x,y) \notin \pi(B) \Rightarrow \liminf_{n\to\infty} \frac{1}{n\log 2}\,[-\log m\pi_x^{-1}(C_n(y))] \geq \alpha. \tag{3.24}$$

Applying Billingsley's lemma and (3.23) shows that

$$\dim(F_x) \geq \alpha \qquad \text{for a.e. } x[\mu] \tag{3.25}$$

which proves the claim (3.17).

(ii) Define $F^* \in \mathcal{T}$ by discarding at each stage of the construction the upper left subsquare, i.e.

$$F^* = \left\{ \left(\sum_{n=1}^{\infty} x_n 2^{-n}, \sum_{n=1}^{\infty} y_n 2^{-n} \right) \middle| \forall n\, (x_n, y_n) \neq (0,1) \right\}. \tag{3.26}$$

F^* is the squared version of the well-known Sierpinski gasket.

For any $x = \sum_1^\infty x_n 2^{-n}$, applying Billingsley's lemma to the measure $m\pi_x^{-1}$ shows

$$\dim(F_x^*) = \liminf_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} x_k. \tag{3.27}$$

Finally, the discussion following formula (3.20) proves (1.4). ∎

Proof of Proposition 3. (i) The only inequality we have to show is

$$\mathrm{Mdim}(F_x) \leq \gamma - 1 \qquad \text{for a.e. } x[m], \tag{3.28}$$

and this is well known to follow from the fact that F is covered by 3^n squares of side 2^{-n}, for every $n \geq 1$. Indeed if we had $\epsilon > 0$ such that

$$m\{x \mid \mathrm{Mdim}(F_x) > \gamma - 1 + 2\epsilon\} = \delta > 0, \tag{3.29}$$

then by Egorov's theorem, for sufficiently large n we would have at least $\frac{1}{2}\delta 2^n$ binary intervals of length 2^{-n} on the x-axis above which F intersects more than $2^{n(\gamma-1+\epsilon)}$ standard binary squares. Since

$$\tfrac{1}{2}\delta 2^{n(\gamma+\epsilon)} > 3^n \qquad \text{for } n \text{ large enough}, \tag{3.30}$$

assumption (3.29) implies a contradiction.

(ii) Fix any $x \in [0,1]$. If $F \in \mathfrak{T}$ is selected randomly, the fibre F_x may be considered as a random Cantor set constructed as follows. Partition the unit interval in two halves; with probability $\frac{1}{4}$ keep only the left half, with probability $\frac{1}{4}$ keep only the right, and with probability $\frac{1}{2}$ keep both. Repeat the same operation in each of the remaining halves, etc. Basic facts about branching processes [AN, chapter 1] imply that $\mathrm{Mdim}(F_x) = \gamma - 1$ almost surely. The result for Hausdorff dimension is slightly harder, but is contained in a much more general theorem of Mauldin and Williams [MW]. An application of Fubini completes the proof of (ii).

(iii) The same F^* employed in Theorem 2(ii) (see (3.26)) will do. The verification is immediate.

(iv) First we construct $F^{(1)} \in \mathfrak{T}$ such that

$$\dim(F_x^{(1)}) = \tfrac{1}{2}, \qquad \mathrm{Mdim}(F_x^{(1)}) = \tfrac{1}{2}[(\gamma - 1) + \tfrac{1}{2}] \tag{3.31}$$

for a.e. $x[m]$.

Start by discarding the upper right subsquare $(\frac{1}{2},1] \times (\frac{1}{2},1]$. We concentrate now on the two left squares, S_t (the top left) and S_b (bottom left). Pick a rapidly increasing sequence of integers $\{n_j\}_{j=1}^{\infty}$. For the first n_1 "generations" of the construction (i.e. until squares of side 2^{-n_1} are discarded) discard squares in S_t randomly, as in (ii), and in S_b always discard the upper left subsquare as in (iii).

For the next n_2 generations reverse the procedure, discarding squares in S_t as in (iii) and in S_b as in (ii). Continue in the same manner, with the strategy between generations $1 + \sum_{i=1}^{k-1} n_i$ and $\sum_{i=1}^{k} n_i$ determined by the parity of k. If $n_{i+1}/n_i \overset{i\to\infty}{\to} \infty$, then standard properties of Hausdorff dimension ([B],[Caj]) and branching processes ([AN]) show that for a.e. $x \in [0,\frac{1}{2}]$, (3.31) holds. The lower right square $[\frac{1}{2},1] \times [0,\frac{1}{2}]$ is treated exactly like the whole unit square, i.e. start by discarding *its* upper right square, etc. This gives (3.31) for a.e. $x \in [0,1]$. Now, instead of dividing each vertical fibre in two and switching the constructions of (ii),

(iii) repeatedly between the two halves, we partition each fibre into 2^k segments. On one of them the construction proceeds as in (iii), on all the rest as in (ii). The location of the distinguished segment is rotated cyclically among the 2^k possibilities: for the first n_1 generations it is the first segment, then for n_2 generations the second, etc.

In this way a set $F^{(k)} \in \mathfrak{T}$ is created for which

$$\text{(3.32)} \qquad \dim(F_x^{(k)}) = \tfrac{1}{2}, \qquad \operatorname{Mdim}(F_x^{(k)}) = 2^{-k}\cdot\tfrac{1}{2} + (1-2^{-k})\cdot(\gamma - 1)$$

for a.e. $x \in [0,1]$.

To get the full power of (iv) increase gradually the value of k in the above construction, for instance proceeding as in $F^{(k)}$ between generations $1 + \sum_{j=1}^{2^k} n_j$ and $\sum_{j=1}^{2^{k+1}} n_j$. ■

Example. The set F^* appearing in Proposition 3(iii) satisfies $\dim(F_x^*) \le \frac{1}{2}$ for a.e. $x \in [0,1]$, but definitely not for *all* x. We now construct $F' \in \mathfrak{T}_0$ which does.

Let $\{n_j\}_{j=1}^\infty$ be a rapidly increasing sequence of integers, and let $\{k_j\}_{j=1}^\infty$ be a sequence in which every positive integer appears infinitely often. Specifically, we require

$$\text{(3.33)} \qquad \frac{n_{j+1}}{n_j} \to \infty, \qquad n_j > k_j.$$

Between generations $n_j + 1$ and n_{j+1} in the construction of F', discard the upper subsquare corresponding to the k_j-th bit y_{k_j} of the y coordinate in a square (this bit is constant on a binary square of side 2^{-n} for $n > n_j$, by (3.33)). Precisely, discard the upper right subsquare if $y_{k_j} = 1$, and the upper left subsquare otherwise. This defines F'.

Fix *any* $x = \sum_1^\infty x_n 2^{-n}$. For any $k \ge 1$, there exists $\epsilon = \epsilon(x,k) \in \{0,1\}$ such that

$$\text{(3.34)} \qquad \liminf_{\substack{j\to\infty \\ k_j = k}} \frac{1}{n_{j+1}} \sum_{n=1}^{n_{j+1}} 1_{\{x_n = \epsilon\}} \le \tfrac{1}{2}.$$

Now for all $y = \sum_{k=1}^\infty y_k 2^{-k} \in F_x'$ except one point, $y_k = \epsilon(x,k)$ for some $k \ge 1$, so it follows from well-known properties of Hausdorff dimension [Caj] that $\dim(F_x') \le \frac{1}{2}$ for *all* $x \in [0,1]$.

Remarks. (1) It is quite easy to construct a set $F \in \mathfrak{T}$ such that

$$\text{(3.35)} \qquad \dim(F_x) = \operatorname{Mdim}(F_x) = \frac{\log(3/2)}{\log 2} \qquad \text{for all } x \text{ except one point.}$$

We conjecture that (3.35) holds for *almost all* $F \in \mathcal{T}$ in the sense of Proposition 3(ii). We can verify this for Minkowski dimension.

(2) Projections of random sets $F \in \mathcal{T}$ (and generalizations) are studied in the papers of Dekking and Grimmett [DG] and Falconer [Fa].

§4. Colored trees

The trees we consider are connected graphs on a countable vertex set $V = \bigcup_{n=0}^{\infty} V_n$ (disjoint union) such that $V_0 = \{v_0\}$ and every $v \in V_{n+1}$ is connected to a unique $\tilde{v} \in V_n$. We say v is a son of $\tilde{v}$ and call v_0 the root. Our trees will be subtrees of the 3-tree T_3 for which every vertex has precisely 3 sons.

DEFINITIONS. (1) Let $\lambda > 0$. The λ-*walk* [L] on a tree T is the Markov chain $\{Y_n\}_{n=0}^{\infty}$ with transition probabilities

$$(4.1) \qquad P[Y_{n+1} = \tilde{w} \mid Y_n = w] = \frac{\lambda}{\lambda + d}, \quad P[Y_{n+1} = w_i \mid Y_n = w] = \frac{1}{\lambda + d} \qquad (1 \le i \le d)$$

where $\{w_i\}_{i=1}^{d}$ are the sons of w, and $\tilde{w}$ the father. In particular, $\lambda = 1$ gives the nearest-neighbor symmetric random walk on T.

(2) A 2-*coloring* χ of T_3 is a function from the vertex set of T_3 to $\{0,1\}$. A *restricted* 2-coloring is a 2-coloring for which both colors appear among the sons of every vertex.

(3) Given a 2-coloring χ of T_3, any sequence $\epsilon = \{\epsilon_n\}_{n=1}^{\infty} \in \{0,1\}^{\mathbf{N}}$ determines a subtree $T(\chi,\epsilon)$ as follows. A vertex W of T_3 is in $T(\chi,\epsilon)$ if

$$(4.2) \qquad \chi(w_1) = \epsilon_1, \quad \chi(w_2) = \epsilon_2, \ldots, \chi(w_n) = \epsilon_n$$

where $v_0, w_1, w_2, \ldots, w_{n-1}, w_n = w$ is the unique geodesic connecting the root and w. With these notations, Theorem 1 implies

COROLLARY 6. *Let* $0 < \lambda < \sqrt{2}$ *and let* χ *be any restricted* 2-*coloring of* T_3. *For almost all* $\epsilon \in \{0,1\}^{\mathbf{N}}$ *with respect to product measure* $(\frac{1}{2},\frac{1}{2})^{\mathbf{N}}$, *the* λ-*walk on* $T(\chi,\epsilon)$ *is transient. The constant* $\sqrt{2}$ *cannot be replaced by a larger constant.*

PROOF. To every restricted 2-coloring χ of T_3, a fractal $F \in \mathcal{T}_0$ can be associated. Attach the root v_0 to $[0,1]^2$. Assume a vertex w in T_3 is attached to a binary square S and w_1, w_2, w_3 are the sons of w. If $\sum_{i=1}^{3} \chi(w_i) = 1$ discard the upper right subsquare of S, attach the vertex colored "1" to the lower right subsquare and the other two vertices to the remaining subsquares. Deal symmetrically with the case $\sum_{i=1}^{3} \chi(w_i) = 2$, and continue inductively. Any color sequence $\{\epsilon_n\}_{n=1}^{\infty}$ which is not eventually constant corresponds to the fibre F_x of F, where $x = \sum_{n=1}^{\infty} \epsilon_n 2^{-n}$. In [L], R. Lyons studied the branching of a tree, a notion closely related to Hausdorff dimension. Theorem 1 implies that for almost all $\epsilon \in \{0,1\}^{\mathbf{N}}$, the tree $T(\chi,\epsilon)$ has branching number $\sqrt{2}$ (in the terminology of [L, §2]). Now [L, theorem 4.3] completes the proof. ■

REMARK. Most of the results in this paper hold for more general constructions than that defining $\mathcal{T}$. For instance, if one deals in the corollary above with restricted l-colorings of the k-tree, the critical constant $\sqrt{2}$ is replaced by $(k-l+1)^{1/l}$. As no new ideas appear in this generalization, we omit it.

PROOF OF PROPOSITION 4. Applying the correspondence in the previous corollary, the "greedy algorithm" appears as follows. Given a 2-coloring χ of T_3, select the color ϵ_1 according to the coloring of most vertices in level 1. If colors $\epsilon_1, \ldots, \epsilon_{n-1}$ have been selected, they determine a subtree $T^{(n-1)}$ of T_3, with n-th level $V(n)$. Now let

$$\epsilon_n = 1 \quad \text{if} \sum_{v \in V(n)} \chi(v) > \frac{1}{2}\,|V(n)|, \quad \text{and} \quad \epsilon_n = 0 \quad \text{otherwise.} \tag{4.3}$$

Continuing inductively, this defines a sequence $\epsilon = \{\epsilon_n\}_{n=1}^{\infty} \in \{0,1\}^{\mathbf{N}}$. By construction, the n-th level of $T(\chi,\epsilon)$ has at least $(\frac{3}{2})^n$ vertices.

We now define a restricted 2-coloring χ_0 as follows. $\chi_0(v_0)$ is arbitrary. Assume χ_0 has been defined already on levels $V_0, V_1, \ldots, V_n$ of T_3. The greedy algorithm selects accordingly a color sequence $\epsilon_1, \ldots, \epsilon_n$, which determines a subtree $T^{(n)}$ of T_3. Denote by a_n the number of vertices in level n of $T^{(n)}$, and assume they are numbered $1, \ldots, a_n$. For each vertex with index $\le \frac{1}{2}a_n + 1$, color two of its sons "0" and the other "1". Reverse this for the other vertices.

In the next level, number the vertices so that vertices of lower index in level n have sons of lower index in level $n+1$, and continue inductively.

For the resulting coloring χ_0, the greedy algorithm selects a sequence $\{\epsilon_n\}_1^{\infty} = \epsilon$ of colors. It is easily seen that the tree $T(\chi_0,\epsilon)$ has only countably many infinite geodesic rays emanating from its root (compare [BP, §5, example 1]). Utilizing the correspondence in Corollary 6 completes the proof.

EXAMPLE. There exists a (non-restricted) 2-coloring χ_1 of T_3 such that:

(i) At any level, at most $\frac{2}{3}$ of the vertices are colored in the same color.

(ii) For any $\epsilon \in \{0,1\}^{\mathbf{N}}$, the tree $T(\chi_1,\epsilon)$ consists of at most a unique infinite geodesic ray of T_3.

CONSTRUCTION (SKETCH). Define χ_1 inductively so that, for any binary sequence $(\epsilon_1,\epsilon_2,\ldots,\epsilon_{2k})$, the subtree of T_3 it determines consists of a geodesic segment $v_0,w_1,w_2,\ldots,w_k$ of length k, followed by a copy of a subtree of T_3, rooted at w_k. ∎

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